# Elimination ideals and Bezout relations

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# FEW REMARKS ON IDEALS

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## 1. Introduction

Let $I \subset \mathbb{K}[x_1, ..., x_n]$ be an ideal such that dim $V(I) = q$. Using Hilbert Nullstellensatz we can easily see, that in the set $I \cap \mathbb{K}[x_1, ..., x_{q+1}]$ there exist non-zero polynomials. It is interesting to know the degree of the minimal polynomial of this type. Here using ideas from [1] we get a sharp estimate for the degree of such minimal polynomial in terms of degrees of generators of the ideal $I$. In fact, in general we solve this problem only for fields of characteristic zero.

## 2. Main Result

Let us recall (see [1]):

**Theorem 1.** *(Perron Theorem) Let $\mathbb{L}$ be a field and let $Q_1, \ldots, Q_{n+1} \in \mathbb{L}[x_1, \ldots, x_m]$ be non-constant polynomials with $\deg Q_i = d_i$. If the mapping $Q = (Q_1, \ldots, Q_{n+1}) : \mathbb{L}^n \to \mathbb{L}^{n+1}$ is generically finite, then there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in \mathbb{L}[T_1, \ldots, T_{n+1}]$ such that*

*(a)* $W(Q_1, \ldots, Q_{n+1}) = 0$,

*(b)* $\deg\, W(T_1^{d_1}, T_2^{d_2}, \ldots, T_{n+1}^{d_{n+1}}) \leq \prod_{j=1}^{n+1} d_j$.

and (see [1]):

**Lemma 2.** *Let $\mathbb{K}$ be an infinite field. Let $X \subset \mathbb{K}^m$ be an affine algebraic variety of dimension $n$. For sufficiently general numbers $a_{ij} \in \mathbb{K}$ the mapping*

$$\pi : X \ni (x_1, \ldots, x_m) \to \Big(\sum_{j=1}^{m} a_{1j}x_j, \sum_{j=2}^{m} a_{2j}x_j, \ldots, \sum_{j=n}^{m} a_{1j}x_j\Big) \in \mathbb{K}^n$$

*is finite.* $\square$

**Theorem 3.** *Let $\mathbb{K}$ be an algebraically closed field and let $f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \geq d_2 ... \geq d_s$. Assume that $I = (f_1, \ldots, f_s) \in \mathbb{K}[x_1, \ldots, x_n]$ is an ideal, such that $V(I)$ has dimension $q$. If we take a sufficiently general system of coordinates $(x_1, \ldots, x_n)$, then there exist polynomials $g_j \in \mathbb{K}[x_1, \ldots, x_n]$ and a non-zero polynomial $\phi(x) \in \mathbb{K}[x_1, ..., x_{q+1}]$ such that*

*(a)* $\deg g_j f_j \leq d_s \prod_{i=1}^{n-q-1} d_i$,

*(b)* $\phi(x) = \sum_{j=1}^{k} g_j f_j$.

*Proof.* Take $F_{n-q} = f_s$ and $F_i = \sum_{j=i}^{s} \alpha_{ij} f_j$ for $i = 1, ..., n-q-1$, whre $\alpha_{ij}$ are sufficiently general. Take $J = (F_1, ..., F_{n-q})$. Then $\deg F_{n-q} = d_s$ and $\deg F_i = d_i$ for $i = 1, ..., n-q-1$. Moreover, $V(J)$ has pure dimension $q$ and $J \subset I$. The mapping

$$\Phi : \mathbb{K}^n \times \mathbb{K} \ni (x, z) \to (F_1(x)z, \ldots, F_{n-q}(x)z, x) \in \mathbb{K}^{n-q} \times \mathbb{K}^n$$

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is a (non-closed) embedding outside the set $V(J) \times \mathbb{K}$. Take $\Gamma = \mathrm{cl}(\Phi(\mathbb{K}^n \times \mathbb{K}))$. Let $\pi : \Gamma \to \mathbb{K}^{n+1}$ be a generic projection. Define $\Psi := \pi \circ \Phi(x, z)$. By Lemma 2 we can assume that

$$\Psi = (\sum_{j=1}^{n-q} \gamma_{1j} F_j z + l_1(x), \ldots, \sum_{j=n-q}^{n-q} \gamma_{nj} F_j z + l_n(x), l_{n-q+1}(x), ..., l_{n+1}(x)),$$

where $l_1, \ldots, l_{n+1}$ are generic linear form. In particular we can assume that $l_{n-q+i}$, $i = 1, .., q+1$ is the variable $x_i$ in a new generic system of coordinates.

Apply Theorem 1 to $\mathbb{L} = \mathbb{K}(z)$, the polynomials $\Psi_1, \ldots, \Psi_{n+1} \in \mathbb{L}[x]$. Thus there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in \mathbb{L}[T_1, \ldots, T_{n+1}]$ such that

$$W(\Psi_1, \ldots, \Psi_{n+1}) = 0 \text{ and } \deg W(T_1^{d_1}, T_2^{d_2}, \ldots, T_k^{d_k}, T_{k+1}, \ldots, T_{n+1}) \le d_s \prod_{j=1}^{n-q-1} d_j,$$

where $k = n - q$. Since the coefficients of $W$ are in $\mathbb{K}(z)$, there is a non-zero polynomial $\tilde{W} \in \mathbb{K}[T_1, \ldots, T_{n+1}, Y]$ such that

(a) $\tilde{W}(\Psi_1(x, z), \ldots, \Psi_{n+1}(x, z), z) = 0$,

(b) $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, \ldots, T_k^{d_k}, T_{k+1}, \ldots, T_{n+1}, Y) \le d_s \prod_{j=1}^{n-q-1} d_j$, where $\deg_T$ denotes the degree with respect to the variables $T = (T_1, \ldots, T_{n+1})$.

Note that the mapping $\Psi = (\Psi_1, \ldots, \Psi_{n+1}) : \mathbb{K}^n \times \mathbb{K} \to \mathbb{K}^{n+1}$ is finite outside the set $V(J) \times \mathbb{K}$. Let $\phi' = 0$ describes the image of the projection

$$\pi : V(J) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}$$

(recall that we consider generic system of coordinates).

The set of non-properness of the mapping $\Psi$ is contained in the hypersurface $S = \{T \in \mathbb{K}^{n+1} : \phi'(T) = 0\}$. Since the mapping $\Psi$ is finite outside $S$, for every $H \in \mathbb{K}[x_1, \ldots, x_n, z]$ there is a minimal polynomial $P_H(T, Y) \in \mathbb{K}[T_1, \ldots, T_{n+1}][Y]$ such that $P_H(\Psi_1, \ldots, \Psi_{n+1}, H) = \sum_{i=0}^{r} b_i(\Psi_1, \ldots, \Psi_{n+1}) H^{r-i} = 0$ and the coefficient $b_0$ satisfies $\{T : b_0(T) = 0\} \subset S$. Now set $H = z$.

We have

$$\deg_T P_z(T_1^{d_1}, T_2^{d_2}, \ldots, T_n^{d_n}, T_{n+1}, Y) \le d_s \prod_{j=1}^{n-q-1} d_j$$

and consequently we obtain the equality $b_0(x_1, ..., x_{q+1}) + \sum F_i g_i = 0$, where $\deg F_i g_i \le \prod_{j=1}^{n-q} d_j$. Set $\phi = b_0$. By the construction the polynomial $\phi$ has zeros only on the image of the projection

$$\pi : V(J) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}.$$

□

**Remark 4.** Simple application of the Bezout theorem shows that our estimations on the degree of $\phi$ is sharp.

**Corollary 5.** *Let $I$ be as above. If $V(I)$ has pure dimension $q$ and $I$ has not embedded components, then there is a polynomial $\phi_1 \in \mathbb{K}[x_1, ..., x_{q+1}]$ which describes the image of the projection*

$$\pi : V(I) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}$$

*such that*

*(a)* $\phi_1 \in I$,

*(b)* $\deg \phi_1 \le d_s \prod_{i=1}^{n-q-1} d_i$.

*Proof.* Let $I = \bigcap^r I_k$ be a primary decomposition of $I$. Then dim $V(I_k) = q$ for every $k$. Let $\phi$ be a polynomial as above. If $\phi = \phi_1\phi_2$, where $\phi_2$ does not vanish on any component of $V(I)$ then $\phi_1 \in I_k$ for every $k$ (by properties of primary ideals) and consequently $\phi_1 \in I$. But $\phi_1$ describes the image of the projection

$$\pi : V(I) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}.$$

$\square$

**Theorem 6.** *Let $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \ge d_2... \ge d_s$. Let $I = (f_1, ..., f_s)$ be the ideal in $\mathbb{C}[x_1, ..., x_n]$ such that $\dim V(I) = q$. Then in $I$ we can find a non-zero polynomial $\phi(x) \in \mathbb{C}[x_1, ..., x_{q+1}]$ such that $\deg \phi \le d_s \prod_{i=1}^{n-q-1} d_i$.*

*Proof.* By theorem 3 for generic $\alpha = (\alpha_{i,j};\ i = 1, ..., q+1, j \ge i)$ there exist a non-zero polynomial $\phi_\alpha \in \mathbb{C}[t_1, ..., t_{q+1}]$ such that

a) $\deg \phi_\alpha \le d_s \prod_{i=1}^{n-q-1} d_i$,

b) $\phi_\alpha(X_1(\alpha), X_2(\alpha), ..., X_{q+1}(\alpha)) \in I$, where $X_1(\alpha) = \alpha_{1,1}x_1 + ... + \alpha_{1,n}x_n$, $X_2(\alpha) = \alpha_{2,2}x_1 + ...\alpha_{2,n}x_n$, ..., $X_{q+1}(\alpha) = \alpha_{q+1,q+1}x_{q+1} + ... + \alpha_{q+1,n}x_n$.

For a polynomial $p = \sum_\alpha a_\alpha x^\alpha \in \mathbb{C}[x_1, ..., x_n]$ we define a norm $||p|| = max_\alpha |a_\alpha|$. In particular for every generic $\alpha$ we can assume that $||\phi_\alpha|| = 1$ (we consider the polynomial $\frac{\phi}{||\phi||}$ instead of $\phi$). Moreover, we can take generic $\alpha_m$ in this way that $X_i(\alpha_m) \to x_i$ for $m \to \infty$ and for $i = 1, ..., q+1$. Polynomial $\phi_\alpha$ we can treat as an element of a vector space $B(D)$ of all polynomials from $\mathbb{C}[x_1, ..., x_n]$ of degree bounded by $D = d_s \prod_{i=1}^{n-q-1} d_i$. Since the norm of every $\phi_{\alpha_m}$ is bounded by 1, we can assume that this sequence converges to a polynomial $\phi$, with norm 1 and of degree bounded by $D = d_s \prod_{i=1}^{n-q-1} d_i$. Thus also a sequence $\phi_{\alpha_m}(X_1(\alpha_m), ..., X_{q+1}(\alpha_m))$ tends to the same polynomial $\phi$. Of course it is non-zero because $||\phi|| = 1$. It is enough to show that $\phi \in I$. However the space $I \cap B(D)$ is a linear subspace of a finitely dimensional complex vector space $B(D)$, hence it is closed subset of $B(D)$. This finishes the proof. $\square$

**Remark 7.** By Lefchetz Principle Theorem 6 holds for every field of characteristic zero.

**Corollary 8.** *Let $f_1, \ldots, f_s \in \mathbb{C}[x_1, ..., x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \ge d_2... \ge d_s$. Let $I = (f_1, ..., f_s)$ be the ideal in $\mathbb{C}[x_1, ..., x_n]$ such that $\dim V(I) = 1$. Assume that the image $S$ of the projection $x \mapsto (x_1, x_2)$ of a one dimensional part of $V(I)$ into $\mathbb{C}^2$ is a curve. Then in $I$ we can find a non-zero polynomial $\phi(x_1, x_2) = \phi_1(x_1, x_2)\rho(x_1) \in \mathbb{C}[x_1, x_2]$ such that*

*a)* $\deg \phi \le D^2 - 2D + 2$, *where* $D = d_s \prod_{i=1}^{n-2} d_i$,

*b) $\phi_1$ describes the image of the projection $x \mapsto (x_1, x_2)$ of a one dimensional part of $V(I)$ into $\mathbb{C}^2$.*

*Proof.* Consider the ideal $I_\alpha = (f_s, \sum_{i=1}^{s-1} \alpha_{1i} f_i, \sum_{i=2}^{s-1} \alpha_{2i} f_i, ..., \sum_{i=n-2}^{s-1} \alpha_{2i} f_i)$. By Theorem 6 we have that there exists a non-zero polynomial $\phi_\alpha \in I \cap \mathbb{C}[x_1, x_2]$ of degree bounded by $D$. We can write $\phi_\alpha = \phi_{1,\alpha}\phi_{2,\alpha}$, where $\phi_{1,\alpha}$ describes $S$ and $\phi_{2,\alpha}$ does not vanish on $S$. Since the degree of $\phi_{1,\alpha}$ is bounded, there is infinitely many $\alpha_i$ such that $\phi_{1,\alpha_i}$ up to a multiplicative constant is the same. Moreover, the ideal $J = (\phi_{2,\alpha_i}, i = 1, 2, ...)$ describes the zero dimensional part of the image of the projection $x \mapsto (x_1, x_2)$ of $V(I)$ into $\mathbb{C}^2$. If deg $\phi_1 = a$, then deg $\phi_{2,\alpha_i} \le D - a$. Consequently we can find a non zero polynomial $\rho(x_1) \in J$, such that deg $\rho \le (D-a)^2$. The ideal $I$ contains a polynomial $\phi\rho$ of degree bounded by $a + (D - a)^2$. The expression $a + (D - a)^2$, where $0 < a \leq D$ attains its maximal value for $a = 1$. This finishes the proof. $\square$

**Remark 9.** If the ideal $I$ has not embedded components, then we can assume that the polynomial $\rho$ describes the image of the projection $x \mapsto x_1$ of a zero dimensional part of $V(I)$ into $\mathbb{C}$. In general case it can describe also some extra points.

## References

[1] Jelonek, Z. *On the Effective Nullstellensatz*. Invent. Math. 162(2005), pp 1-17.